Adiabatic Invariants

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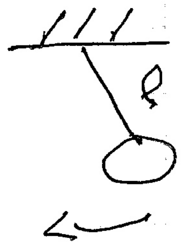
→ 

$$\ell = \ell(t)$$

$$\text{s/t} \quad \boxed{\frac{\dot{\ell}}{\ell} < \sqrt{g/\ell}}$$

→ 2 time scale

→ 'adiabatic' variation of parameter.

→ How describe?

→ Now $\ddot{\theta} + \dfrac{g}{\ell(t)}\theta = 0$

$$\ell(t) = \ell(\epsilon t)$$

$\downarrow$ slow time in parameter variation

$$\ddot{\theta} + \frac{g}{\ell(\epsilon t)}\theta = 0$$

$$\epsilon t = \tau$$

$$\Rightarrow dt = \frac{1}{\epsilon}\, d\tau$$

$$d/dt = \epsilon \frac{d}{d\tau}$$

$$\ddot{\theta} + \frac{g}{\epsilon^2 \ell(\tau)}\theta = 0 \quad \Rightarrow \quad \boxed{\ddot{\theta} + \frac{\omega(\tau)^2}{\epsilon^2}\theta = 0}$$

generically points toward $\underline{\underline{\text{WKB}}}$.

i.e. generic:

$$\ddot{x} + \omega^2(\epsilon t)\, x = 0$$

$\Rightarrow \tau = \epsilon t$

$$\frac{d^2 x}{d\tau^2} + \frac{\omega^2(\tau)}{\epsilon^2}\, x = 0$$

→ A different look at adiabatic theory ....

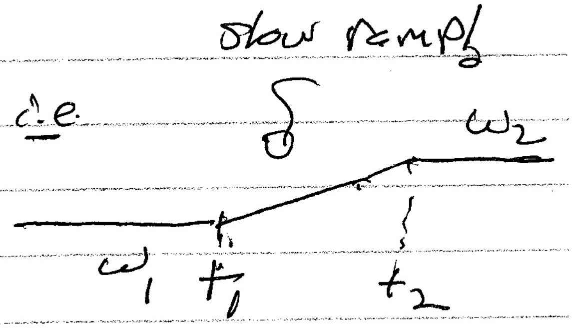
One might forego canonical formalism, and simply investigate an oscillator with slowly varying frequency

i.e.

$$\ddot{x} + \omega^2 x = 0 \quad \Rightarrow$$

$$\ddot{x} + \underbrace{\omega^2(\epsilon t)}_{\text{slowly varying frequency}} x = 0$$

i.e.



i.e.

$$\frac{1}{\omega}\frac{d\omega}{dt} \sim O(\epsilon\omega) \ll 1$$

⇒ expect, on basis of previous discussion,

I ⇒ adiabatic invariant

i.e. if $a \equiv$ oscillator amplitude, then

$$I = E/\omega = \tfrac{1}{2} m\omega^2 a^2/\omega \simeq m\omega a^2$$

as const

to show.

Now, for slowly varying ω, can solve by WKB !

now, $\boxed{\epsilon t = \tau}$

$$\frac{d^2X}{d\tau^2} + \frac{\omega^2(\tau)}{\epsilon^2} X = 0$$

$$X(\tau) = a_0 e^{i\phi(\tau)/\epsilon}$$

where: $\underset{\text{eikonal}}{\phi_0} + \epsilon \underset{\text{correction} \Rightarrow \text{ampl.}}{\phi_1} + \ldots$

$$\frac{d}{d\tau}\left(a_0 \frac{i\dot{\phi}(\tau)}{\epsilon} e^{i\phi(\tau)}\right) + \frac{\omega(\tau)^2}{\epsilon^2} a_0 e^{i\phi(\tau)} = 0$$

$$\left(-\frac{\dot{\phi}^2}{\epsilon^2} + \frac{i\ddot{\phi}(\tau)}{\epsilon}\right) a_0 e^{i\phi} + \frac{\omega(t)^2}{\epsilon^2} a_0 e^{i\phi(\tau)} = 0$$

need to $O(1/\epsilon)$

$\Rightarrow$

$$\left(-\frac{(\dot{\phi}_0 + \epsilon\dot{\phi}_1)^2}{\epsilon^2} + \frac{i\ddot{\phi}_0(\tau)}{\epsilon}\right) + \frac{\omega(\tau)^2}{\epsilon^2} = 0$$

$$-\frac{\dot{\phi}_0^2}{\epsilon^2} + \frac{\omega(\tau)^2}{\epsilon^2} = 0$$

$$\dot{\phi}_0(\mathcal{T}) = \omega(\mathcal{T})$$

$$\phi_0(\mathcal{T}) = \int \omega(\mathcal{T})\, d\mathcal{T}$$

For next order correction

$$-\frac{2\dot{\phi}_0 \dot{\phi}_1}{\epsilon} + \frac{i\ddot{\phi}_0}{\epsilon} = 0$$

$$\dot{\phi}_1 = i\ddot{\phi}_0 / 2\dot{\phi}_0$$

$$= \frac{i}{2}\frac{d}{dt}\ln\left(\dot{\phi}_0(\mathcal{T})\right)$$

$$\phi_1 = \frac{i}{2}\ln\left(\dot{\phi}_0(\mathcal{T})\right)$$

$$= \frac{i}{2}\ln\left(\omega(\mathcal{T})\right)$$

So

$$x(t) = a_0\, e^{i\phi(\mathcal{T})/\epsilon}$$

$$= a_0\, e^{i\int \frac{\omega(\mathcal{T})}{\epsilon} dt}\, e^{i\frac{i}{2}\ln(\omega(\mathcal{T}))}$$

$$= \underline{a_0}\, e^{i\int \frac{\omega(\mathcal{T})}{\epsilon} dt}\, e^{-\frac{\ln}{2}\omega(\mathcal{T})}$$

$$\Rightarrow \quad x(t) = \underline{a_0}\, e^{i\int \frac{\omega(\tau)}{\epsilon} dt}\, e^{-\frac{1}{2}\ln\omega}$$

$$= \frac{a_0}{\sqrt{\omega}}\, e^{i\int \frac{\omega(\tau)}{\epsilon} \cancel{dt}}$$

$$d\tau = \epsilon\, dt$$

re-scaling; $t = \tau/\epsilon$; $\omega(\epsilon t)$

$$\boxed{x(t) = \frac{a_0}{\sqrt{\omega}}\, e^{i\int \omega(\epsilon t)\, dt}}$$

WKB soln.

and can observe: $\overline{} \Rightarrow$ cycle (fast time) average.

$$\overline{\omega x^2} = \omega\, \overline{x^2} = \omega \frac{a_0^2}{\omega} = \text{const!}$$

(cycle) action

$\Rightarrow$ **Action is invariant, due to frequency modulation of amplitude!**

check:

$$I = \frac{1}{2\pi}\oint p\, dq$$

$$= \frac{1}{2\pi}\oint p\, dx$$

$$= \frac{1}{2\pi}\oint m\dot{x}\, dx = \frac{1}{2\pi}\oint m\dot{x}\,\dot{x}\, dt$$

$$I = \frac{1}{2\pi} \oint_{\omega t} m\dot{x}^2 \, dt$$

$$x(t) = \frac{q_0}{\sqrt{\omega}} \cos(\omega t + \delta)$$

$$\dot{x} = -q_0 \sqrt{\omega} \sin(\omega t + \delta)$$

$$\theta = \omega t \qquad \delta \to 0 \qquad \omega^2/(\sqrt{\omega})^2$$
$$d\theta = \omega dt$$

$$I = \frac{1}{2\pi} \oint p\,dq = \frac{1}{2\pi} \oint d\theta\, q_0^2\, \omega \sin^2\theta \frac{d\theta}{\omega}$$

$$= \frac{1}{2} q_0^2 \Rightarrow \text{real const.!}$$

$\Rightarrow$ the message:

- adiabatic invariants basically a consequence of WKB approximation (time scale sep!) of time scale separation via

- $\boxed{\text{WKB would lead one to adiabatic invariance of action, even if did not realize it}}$

- need retain WKB correction beyond pure eikonal for freq. modulation of amplitude $\Rightarrow$ essential!

→ Adiabatic Invariants [and Action-Angle Variables]

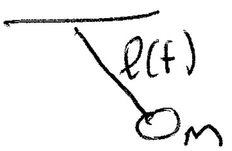
i) Adiabatic Invariants

(Bounded phase space ⟹ QPO)

→ Consider finite motion in 1D. Motion characterized by λ parameter, such that:

$$\frac{1}{\lambda}\frac{d\lambda}{dt} \ll \frac{1}{\mathcal{T}}$$

↳ period of motion

i.e. 

$$\frac{1}{\ell}\frac{d\ell}{dt} < \sqrt{g/\ell}$$

pull on string

thus, $\dot{E}$ will be "small / slow" (i.e. $H = H(\lambda(t), p, q)$

parametric dependence)

Now, $$\frac{dE}{dt} = \frac{\partial H}{\partial t} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}$$

as λ varies slowly compared to $\omega_0 = 1/\mathcal{T}$, can average over t on fast scales, i.e.

$$\frac{d\bar{E}}{dt} = \overline{\frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}} \cong \overline{\frac{\partial H}{\partial \lambda}}\,\frac{d\lambda}{dt}$$

↳ avg. over motion q, p → fast

{break avg. on basis time scale separation where $\overline{A} = \frac{1}{T}\int^{\tau} dt\, A(t)$ → holding E fixed!

average on fast timescale

essential.

⇒

$$\overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T}\int_0^{\tau} \frac{\partial H}{\partial \lambda}\, dt$$

Now,

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$dt = \int dq / \partial H/\partial p$$

∴ can take $\int dt \rightarrow \oint \frac{dq}{\partial H/\partial p}$

$\oint$ ↔ complete circuit orbit.

so finally,

$$\frac{d\overline{E}}{dt} = \frac{d\lambda}{dt}\left\{\frac{\oint (\partial H/\partial \lambda)\, dq / (\partial H/\partial p)}{\oint dq/(\partial H/\partial p)}\right\}$$

avg

$$\equiv \frac{d\lambda}{dt}\left\langle \frac{\partial H}{\partial \lambda}\right\rangle_{\tau}.$$

per

Now: – integrations must be performed for fixed, given value of λ (i.e. $\dot{\lambda}/\lambda \ll \omega_0$)

– on such path (n.b. why "path" of interest!), $H \equiv E$ and $p = p(q; E, \lambda)$

$$\therefore \quad H(p, q, \lambda) = E$$

{path for E const.

$$\frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p}\frac{\partial p}{d\lambda} = 0$$

$$\Rightarrow \quad \frac{\partial H/\partial \lambda}{\partial H/\partial p} = -\frac{\partial p}{\partial \lambda}$$

plug in previous

$$\therefore \quad \frac{d\bar{E}}{dt} = \frac{d\lambda}{dt}\,\frac{\oint -(\partial p/\partial \lambda)\,dq}{\oint dq\,\partial p/\partial E}$$

$(1/\partial H/\partial p = \partial p/\partial E)$ (fixed λ)

so, re-writing:

$$\frac{d\bar{E}}{dt}\oint dq\,\partial p/\partial E + \frac{d\lambda}{dt}\oint(\partial p/\partial \lambda)\,dq = 0$$

$$\Rightarrow \oint_{\substack{E,\lambda \\ \text{fixed}}} dq \left\{ \frac{\partial p}{\partial E}\frac{dE}{dt} + \frac{\partial p}{\partial \lambda}\frac{d\lambda}{dt} \right\} = 0$$

$$\Rightarrow \boxed{\overline{dI/dt} = 0}$$

Note:
$I = S = \oint p\,dq$

where $\boxed{I = \oint_{\substack{E,\lambda \\ \text{fixed}}} \frac{p\,dq}{2\pi}}$ → $\left\{\right.$ integral taken over path for fixed, given E, λ !

∴ → $\boxed{\begin{array}{l} I \text{ const. as } \lambda \text{ varies !} \\ \therefore\ I \text{ adiabatic invariant} \end{array}}$

→ in general (including higher dimensions)

$$I_c = \oint_\gamma \underline{p} \cdot d\underline{q} = \iint_\sigma d\underline{p} \wedge d\underline{q} \qquad \left\{ \begin{array}{l} \text{Liouville} \\ \text{Thm,} \\ \text{again} \end{array} \right.$$

is Poincaré's relative integral invariant
(γ closed curve, enclosing σ)

I_c is exact invariant

$$I = \oint p \, dq$$

so $I = I_c \Big|_{E,\,\lambda \text{ constant}}$

is approximation to Poincare invariant

for $\dot{\lambda}/\lambda < \omega_0$.
long time scales

{ Hence adiabatic invariant.

Now, adiabatic invariant:

$$I = \oint_{\lambda, E \text{ fixed}} dq \, p/2\pi \quad \to \text{what is it?}$$

so $I = I(E)$

$$= \oint \frac{p\,dq}{2\pi}$$

$$2\pi \frac{\partial I}{\partial E} = \oint \frac{\partial p}{\partial E} dq = \oint \frac{dq}{\partial H/\partial p} = \tau, \text{ by Hamilton's Eqns}$$

$$\therefore \boxed{\partial I/\partial E = 1/\omega}$$

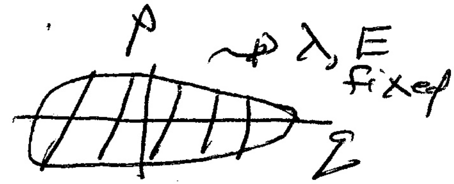
$$\boxed{\partial E/\partial I = \omega}$$

Now, of course:

adiabatic invariant has geometrical significance.

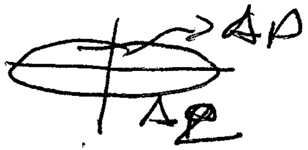
$$I = \oint_{E,\lambda} \frac{p\,dq}{2\pi} = \iint_{E,\lambda} \frac{dp\,dq}{2\pi}$$

i.e.



∴ I corresponds to enclosed area!

e.g. $$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2 = E$$



$$\Delta p = (2mE)^{1/2}$$
$$\Delta q = (2E/m\omega^2)^{1/2}$$

$$\text{Area} = \pi\, \Delta q\, \Delta p = 2\pi E/\omega$$

$$\boxed{I = E/\omega}$$ → for oscillator adiabatic invariant is action, E/ω.

∴ $\dot{\ell}/\ell < \omega_0 \Rightarrow$

$$E \sim \omega \sim \sqrt{g/\ell}$$

Physics 200 A, B Mechanics

→ Adiabatic Invariants: Review

→ if $H = H(p, q, \lambda(t))$

$\lambda(t)$ → parametric dependence

with a) periodic motion for fixed λ,

b.) $\frac{1}{\lambda}\frac{d\lambda}{dt} < \omega$

$\frac{1}{\lambda}\frac{d\lambda}{dt}$ → rate of change of parameter

ω → motion frequency

then $I_\lambda = \oint_{C_\lambda} \underline{p} \cdot d\underline{q}$ ≡ action computed at fixed value of λ is adiabatic invariant

→ adiabatic invariant is COM on time scales $\tau > \omega^{-1}$.

→ adiabatic invariant is intrinsically / implicitly referenced to a given time scale. Same system can manifest multiple adiabatic invariants on different time scales.

→ $I = \oint_C \underline{p} \cdot d\underline{q}$ → Poincare - Cartan Invariant

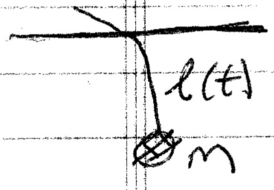
→ exact COM

to calculate explicitly, need integrable motion (as in explicit representation of action-angle var.)

but:

— $I_\lambda = \oint_{C_\lambda} \underline{p}\cdot d\underline{q}$ is approximation to I, computed for fixed λ. $\dot{I}_\lambda \cong 0$ for $t \gg \omega^{-1}$.

Examples: 1) Pendulum — the Prototype


$\ell(t)$, m

$$\frac{\dot{\ell}(t)}{\ell} < \sqrt{g/\ell}\,.$$

How does Θ vary with ℓ?

$$I = E/\omega \quad , \text{ understood } E = \overline{E}$$

$$\omega = \sqrt{g/\ell}$$

$$\overline{E} = \frac{m}{2}\ell^2\overline{\dot{\Theta}^2} + mg\ell\,\frac{\overline{\Theta^2}}{2}$$

$$= mg\ell\,\overline{\Theta^2}$$

$$I = m\sqrt{g}\,\ell^{3/2}\,\overline{\Theta^2}$$

so $\Theta_{rms} \sim \ell^{-3/4}$ i.e. amplitude decreases as length increases